Non-monotonic Logic I

Bridges between classical and non-monotonic consequences

Michal Peliš*

1 Common reasoning

monotonicity

$$\frac{\Gamma \mid\!\sim \varphi}{\Gamma \cup \Delta \mid\!\sim \varphi}$$

can fail

caused by: background knowledge, implicit facts, presuppositions, etc.

An example of common reasoning

```
Smith entered the office of his boss.
He was nervous.
```

1. **typically**: (x enters the office of his boss) $\mid\!\sim$ (x is nervous)
2. **infer**: *Smith is nervous.* (*)
3. **new info**: `After all, he did not want to lose his best employee.`
4. **then**: (*) is wrong

1.1 Where to use nonmonotonic reasoning

- legal reasoning
- diagnosis
- natural language understanding
- intelligent tutoring systems

*Main theorems, basic notions, and exercises for the course *Non-monotonic logic/Theory of rational reasoning* at the Dpt of logic, Faculty of Philosophy, Charles University in Prague.

1.2 Ways of getting more

1. new assumptions
2. restricting the set of models and preferential relations among models
3. new rules

2 Classical consequence

Consequence relation is a set of pairs $\langle \Gamma, \Delta \rangle$ where Γ and Δ are sets of formulas (Γ is a set of premisses). We will use the version $\langle \Gamma, \{\varphi\} \rangle$ where φ is a formula.

Language of classical propositional logic $\mathcal{L}_{\mathsf{CPL}}$ with a subset of signs for atomic formulas $\mathcal{A} = \{p, q, \ldots\}$ and formulas defined as follows:

$$\varphi := p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \psi_1 \rightarrow \psi_2 \mid \psi_1 \leftrightarrow \psi_2$$

Formal system based on language of classical propositional logic: CPL.

- *syntactical* consequence relation ($\Gamma \vdash \varphi$) over CPL

$$\psi_0, \psi_1, \ldots, \psi_n = \varphi$$

  - $\psi_i \in \Gamma$
  - $\psi_i \in \mathsf{CPL}$
  - ψ_i is a result of an application of a rule to some $\psi_k, \ldots, \psi_l$ (where $k, \ldots, l \leq i$)

- *semantical* consequence relation ($\Gamma \models \varphi$)

(Strong) completeness theorem: $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$

For classical consequence relation (operation) we write $\vdash$ (Cn). $\mathsf{Cn}\Gamma = \{\varphi : \Gamma \models \varphi\}$ or $\{\varphi : \Gamma \vdash \varphi\}$.

2.1 Horn rules

reflexivity (inclusion) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
resp. $\Gamma \subseteq \mathsf{Cn}\Gamma$

cumulative transitivity (Cut) If $\Gamma \vdash \psi$, for each $\psi \in \Delta$, and $\Gamma \cup \Delta \vdash \varphi$, then $\Gamma \vdash \varphi$;
resp. if $\Gamma \subseteq \Delta \subseteq \mathsf{Cn}\Gamma$, then $\mathsf{Cn}\Delta \subseteq \mathsf{Cn}\Gamma$.

monotony If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;
resp. if $\Gamma \subseteq \Delta$, then $\mathsf{Cn}\Gamma \subseteq \mathsf{Cn}\Delta$.

Exercise 1 Plain transitivity (if $\Gamma \vdash \psi$, for each $\psi \in \Delta$, and $\Delta \vdash \varphi$, then $\Gamma \vdash \varphi$) *is equivalent to Cut. Prove it.*

Hint: use reflexivity and monotony.

Exercise 2 Idempotence ($\mathsf{Cn}\Gamma = \mathsf{Cn}(\mathsf{Cn}\Gamma)$) *is equivalent to Cut. Prove it.*

Hint: use reflexivity and monotony.

Exercise 3 *Give some examples of logics where $\alpha \vdash \varphi$ implies $\{\alpha, \beta\} \vdash \varphi$ (singleton monotony), but $\alpha \vdash \varphi$ does not imply $(\alpha \wedge \beta) \vdash \varphi$.*

2.2 Compactness

Γ is satisfiable whenever each finite subset $G \subseteq \Gamma$ is satisfiable.

2.2.1 Maximalizability property

If $\Gamma \nvdash \varphi$, then there is $\Delta \supseteq \Gamma$ such that $\Delta \nvdash \varphi$ and Δ is *maximal* (i.e., whenever $\Sigma \supset \Delta$, then $\Sigma \vdash \varphi$). Every CPL-consistent set can be extended to a maximal CPL-consistent set.

Exercise 4 *Prove that compactness implies maximalizability property.*

3 Supraclassicality

Consequence relation (operation) is *supraclassical* iff

$$\vdash \;\subseteq\; \mid\!\sim$$

($\mathsf{Cn}\Gamma \subseteq \mathsf{C}\Gamma$, for every Γ).

3.1 Paraclassicality

Consequence is *paraclassical* whenever it is supraclassical and satisfies Horn rules (reflexivity, Cut, **monotony**).

3.1.1 Uniform substitution

Uniform substitution is a function $\sigma : \mathcal{A} \longrightarrow Fla$.

Exercise 5 *Define substitution as a function $\sigma : Fla \longrightarrow Fla$ to be homomorphic with respect to logical connectives.*

Exercise 6 *Decide whether*

1. $\varphi \vdash \sigma(\varphi)$,
2. *if φ is tautology, then $\sigma(\varphi)$ is tautology,*

3. if $\Gamma \vdash \varphi$*, then* $\sigma(\Gamma) \vdash \sigma(\varphi)$*,*

for each substitution σ*.*

Theorem 1 *If* $\mid\!\sim$ *is supraclassical, monotonic, and* closed under substitution (if $\Gamma \mid\!\sim \varphi$, then $\sigma(\Gamma) \mid\!\sim \sigma(\varphi)$)*, then* $\mid\!\sim = \vdash$ *or* $\mid\!\sim$ *is total.*

4 Additional background assumptions

Let $K \subseteq Fla$ be a set of background assumptions.

4.1 Pivotal assumptions

Definition 1 (pivotal-assumption consequence) $\Gamma \vdash_K \varphi$ *iff* $\Gamma \cup K \vdash \varphi$.

Exercise 7 *Show that* $\vdash_K$ *is paraclassical and compact consequence relation.*

Exercise 8 *Classical consequence relation satisfies* disjunction in the premisses

$$\frac{\Gamma, \alpha \vdash \varphi \qquad \Gamma, \beta \vdash \varphi}{\Gamma, (\alpha \vee \beta) \vdash \varphi}$$

Does $\vdash_K$ *do the same job?*

Exercise 9 *First, find a counterexample of substitution-closing. Second, what will change if* $\sigma(K)$ *is admitted? Third, let us imagine that* K *includes substitutional instances of all its members, i.e.,* $\sigma(\varphi) \in K$ *for each* $\varphi \in K$*.*

Theorem 2 *Every paraclassical consequence satisfies* left classical equivalence*,* right weakening*, and* free premiss property*.*

left classical equivalence If $\mathsf{Cn}\Gamma = \mathsf{Cn}\Delta$, then $\mathsf{C}\Gamma = \mathsf{C}\Delta$.

right weakening If $\Gamma \mid\!\sim \varphi \vdash \psi$, then $\Gamma \mid\!\sim \psi$.

free premiss property If $\Delta \subseteq \Gamma$, then $\mathsf{C}\Gamma = \mathsf{C}(\mathsf{C}\Delta \cup \Gamma)$.

Exercise 10 *Prove Theorem 2.*

Theorem 3 (Representation theorem) *If* $\mid\!\sim$ *is paraclassical, compact, and satisfies disjunction in the premisses, then there is* $K \subseteq Fla$ *such that* $\mid\!\sim = \vdash_K$*.*

4.2 Default assumptions

We confront our knowledge database K with a gained data Γ for to keep consistency.

Definition 2 *A subset* $K' \subseteq K$ *is* max-consistent *with respect to* Γ *iff* $K' \cup \Gamma$ *is consistent and every* $K'' \supset K'$ *is inconsistent wrt* Γ*.*

Definition 3 (default-assumption consequence) $\Gamma \mid\!\sim_K \varphi$ *iff* $K' \cup \Gamma \vdash \varphi$, *for each* $K' \subseteq K$ *max-consistent wrt* Γ.

$$\mathsf{C}_K\Gamma = \bigcap\{\mathsf{Cn}(K' \cup \Gamma) : K \supseteq K' \text{ max-consistent wrt } \Gamma\}$$

Exercise 11 *If* $K_1, K_2 \subseteq K$ *are max-consistent wrt* Γ *and* $K_1 \subseteq K_2$, *then* $K_1 = K_2$.

Exercise 12 *Show disjunction in the premisses.*

non-compact, non-monotonic

cautious monotony If $\Gamma \mid\!\sim_K \delta$, for each $\delta \in \Delta$, and $\Gamma \mid\!\sim_K \varphi$, then $\Gamma \cup \Delta \mid\!\sim_K \varphi$.

Exercise 13 *Prove that* $\mid\!\sim_K$ *is cautiously monotonic.*

Hint: Prove, first, that if K' is max-consistent wrt Γ and $\Gamma \cup K' \vdash \delta$, then K' is max-consistent wrt $\Gamma \cup \{\delta\}$, for each $K' \subseteq K$.

Exercise 14 $\vdash \subseteq \mid\!\sim_K \subseteq \vdash_K$

Exercise 15 *Prove: if* $K \cup \Gamma$ *is consistent, then* $\mathsf{Cn}(K \cup \Gamma) = \mathsf{Cn}_K\Gamma = \mathsf{C}_K\Gamma$.

syntax dependency for $K \neq \mathsf{Cn}K$; on the other hand:

Theorem 4 *If* $K = \mathsf{Cn}K$ *and* K *is inconsistent wrt* Γ, *then* $\mathsf{C}_K\Gamma = \mathsf{Cn}\Gamma$.

5 Restricting the set of models (valuations)

Let V be a set of all valuations (models). $W \subseteq V$ is a restricted set of valuations.

5.1 Pivotal valuations

Definition 4 (pivotal-valuation consequence) $\Gamma \vdash_W \varphi$ *iff* $(\forall v \in W)(v \models \Gamma \Rightarrow v \models \varphi)$.

non-compact

Definition 5 W *is* definable *set of valuations iff there is* $K \subseteq Fla$ *such that* $W = \{v \in V : v \models K\}$.

Theorem 5 *The pivotal-assumption consequences are precisely the pivotal-valuation consequences determined by a definable* $W \subseteq V$.

Theorem 6 *The pivotal-assumption consequences are precisely the pivotal-valuation consequences that are compact.*

Exercise 16 *Let us have a finite set of atomic formulas* $\{p_1, \ldots, p_n\}$ *that generates formulas in propositional language. Show that the pivotal-assumption consequences are precisely the pivotal-valuation consequences in this special case.*

5.2 Default valuations

We introduce preferences among models (valuations) in W. Let $\succ \subseteq W^2$ be irreflexive and transitive. The couple $\langle W, \succ \rangle$ is called preferential models. We say that $v \in W$ is a minimal model (valuation) for Γ iff $v \models \Gamma$ and $(\forall u \prec v)(u \not\models \Gamma)$; let us write $v \models_{\succ} \Gamma$.

Definition 6 (default-valuation consequence) $\Gamma \mid\!\sim_{W\succ} \varphi$ *iff* $v \models_{\succ} \Gamma$ *implies* $v \models \varphi$.

Exercise 17 $\vdash \subseteq \vdash_W \subseteq \mid\!\sim_{W\succ}$.

Hint: $\vdash_W$ is a special case of $\mid\!\sim_{W\succ}$.

non-monotonic, non-transitive

cautious monotony for well-founded $\succ$-relations

Exercise 18 *Prove that cautious monotony can fail.*

Hint: consider infinite descending chain.

consistency preservation fails

consistency preservation $\Gamma \mid\!\sim \bot \Rightarrow \Gamma \vdash \bot$.

Exercise 19 *Let $\succ$ be well founded. Prove that* $\Gamma \mid\!\sim_{W\succ} \bot \Rightarrow \Gamma \vdash_W \bot$.

Exercise 20 *Prove: if* $W_2 \subseteq W_1 \subseteq V$, *then* $\vdash_{W_1} \subseteq \vdash_{W_2}$. *What about* $\mid\!\sim_{W_1\succ} \subseteq \vdash_{W_2}$*?*

6 Additional rules

Let R be a set of rules: $R = \{\langle \varphi, \psi \rangle : \varphi, \psi \in Fla\}$.
$R(\Gamma) = \{\psi : \langle \varphi, \psi \rangle \in R \text{ and } \varphi \in \Gamma\}$
Γ is closed under R iff $R(\Gamma) \subseteq \Gamma$.

6.1 Pivotal rules

Definition 7 (pivotal-rule consequence) $\Gamma \vdash_R \varphi$ *iff* $\varphi \in \Delta$ *for every* $\Delta \supseteq \Gamma$ *such that* Δ *is closed under* Cn *and* R.

Exercise 21 *Is Fla such a set?*

Exercise 22 *Is* $\vdash_R$ *compact and paraclassical?*

neither *disjunction in the premisses* nor *contraposition*

contraposition If $\varphi \mid\!\sim \psi$, then $\neg\psi \mid\!\sim \neg\varphi$.

Exercise 23 *Show that contraposition is valid for* $\vdash_K$ *and* $\vdash_W$*, but neither for* $\mid\!\sim_K$ *nor for* $\mid\!\sim_{W\succ}$.

Theorem 7 *The pivotal-asumption consequences are precisely the pivotal-rule ones that satisfy disjunction in the premisses.*

Theorem 8 *Let* $\mathcal{K}$ *be a set of all pivotal-assumption consequences,* $\mathcal{W}$ *be a set of all pivotal-valuation consequences, and* $\mathcal{R}$ *be a set of all pivotal-rule consequences. Then* $\mathcal{K} = \mathcal{W} \cap \mathcal{R}$.

For every set of formulas Γ, the following set $\mathsf{Cn}(\Gamma \cup \{(\varphi \to \psi) : \langle\varphi, \psi\rangle \in R\})$ is equal to $\mathsf{Cn}_R\Gamma$ with *disjunction in the premisses.* (Makinson—van der Torre theorem) As a conclusion we get

Theorem 9 $\mathsf{Cn}_R\Gamma \subset \mathsf{Cn}(\Gamma \cup \{(\varphi \to \psi) : \langle\varphi, \psi\rangle \in R\})$*, for every* Γ.

Theorem 10 (Representation theorem) *If* $\mid\!\sim$ *is compact and paraclassical, then there is* R *such that* $\mid\!\sim = \vdash_R$.

6.2 Default rules

The application of a rule is under consistency constraints. Let us suppose, there is an ordering $\succ$ of rules in R.

Definition 8 (default-rule consequence) $\mathsf{C}_{R\succ}\Gamma = \bigcup\{\Gamma_n : n < \omega\}$ *such that*

1. $\Gamma_0 = \mathsf{Cn}\Gamma$
2. *If there is a rule* $\langle\varphi, \psi\rangle \in R$ *such that* $\varphi \in \Gamma_n$ *and* $\psi \notin \Gamma_n$ *and* $\Gamma_n \cup \{\psi\}$ *is consistent, then* $\Gamma_{n+1} = \mathsf{Cn}(\Gamma_n \cup \{\psi\})$*. Otherwise:* $\Gamma_{n+1} = \Gamma_n$.

Exercise 24 $\vdash \subseteq \mid\!\sim_{R\succ} \subseteq \vdash_R$.

References

[1] D. Makinson. *Bridges from Classical to Nonmonotonic Logic.* King's College, 2005.