Non-monotonic Logic II

Default logic

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1 Default reasoning and rules

1.1 Default theories

1.1.1 Default rules

$$d = \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi}$$

- φ—prerequisite
- $\psi_1, \ldots, \psi_n$—justifications ($j(d)$—set of all justifications of d)
- χ—consequent ($c(d)$—consequent of the rule d)

Informal interpretation: If a prerequisite φ is known and all justifications $\psi_1, \ldots, \psi_n$ can be consistently supposed, then conclude χ.

Default theory Let Γ be a set of formulas and D a set of defaults (i.e., default rules). Deafult theory T is a couple (Γ, D).

1.1.2 Operational semantics

Let us form sequences of defaults from D without multiple occurence: $\Pi_1 = \langle d_0, d_1, \ldots \rangle$, Π_2, etc. The ordering of defaults is an order of their application.

For each Π we define:

$$In\Pi = \mathsf{Cn}(\Gamma \cup \{c(d) : d \in \Pi\})$$

$$Out\Pi = \{\neg\psi : (\exists d \in \Pi)(\psi \in j(d))\}$$

Both sets must be understood as arising step by step, i.e., default by default, according to an order in Π.

A default $d = \frac{\varphi : \psi_1, \ldots, \psi_n}{\chi}$ is *applicable* iff φ is included in (so far arised) $In\Pi$ and $\psi_1, \ldots, \psi_n$ are not in a contradiction with (so far arised) $In\Pi$.

*Main theorems, basic notions, and exercises for the course *Non-monotonic logic/Theory of rational reasoning* at the Dpt of logic, Faculty of Philosophy, Charles University in Prague.

Definition 1 Π *is* process *of T iff each $d \in \Pi$ is applicable.*

Definition 2 *Let Π be a process.*

- Π *is* successful *iff $In\Pi \cap Out\Pi = \emptyset$. Otherwise it is* failed.
- Π *is* closed *iff all defaults from D applicable (in the order) are in Π.*

Definition 3 (extension) *A set of formulas E is an* extension *of T iff there is a successful and closed process Π such that $E = In\Pi$.*

Exercise 1 *Let $T = (\Gamma, D)$ be a default theory where $\Gamma = \emptyset$ and $D = \{\frac{:\alpha}{\neg\alpha}\}$. Check whether $\Pi_1 = \langle\rangle$ and $\Pi_2 = \langle\frac{:\alpha}{\neg\alpha}\rangle$ are processes. If so, are they successful and closed?*

Solution: Since all defaults in Π_1 are applicable, Π_1 is a process, but it is not closed because of the applicability of $\frac{:\alpha}{\neg\alpha}$. Let us check this. There is no prerequisite and α is consistent with the 0-step $In\Pi_2 = \mathsf{Cn}\Gamma$. It means that Π_2 is a closed process. After the application of the default we get $In\Pi_2 = \mathsf{Cn}(\Gamma \cup \{\neg\alpha\})$ and $Out\Pi_2 = \{\neg\alpha\}$. Thus, Π_2 is failed. We can conclude that T has no extension.

Exercise 2 *Let $\Gamma = \{\neg p, q\}$ and $D = \{\frac{q:\neg r}{p}\}$. Has $T = (\Gamma, D)$ any extension?*

Exercise 3 *Let $T = (\emptyset, \{\frac{:p}{\neg q}, \frac{:q}{r}\})$. Try to find an extension.*

Hint: There are two closed processes: $\langle\frac{:p}{\neg q}\rangle$ and $\langle\frac{:q}{r}, \frac{:p}{\neg q}\rangle$. Are they successful?

Exercise 4 *Let $\Gamma = \{studentOfPhilFaculty, studentOfLogic\}$ and $D = \{d_0, d_1\}$ where*

- $d_0 = \dfrac{studentOfPhilFaculty : \neg likeMath}{\neg likeMath}$
- $d_1 = \dfrac{studentOfLogic : likeMath}{likeMath}$

Find any extension.

Hint: There are two extensions.

Exercise 5 *Why do we not require $Out\Pi$ to be deductively closed?*

Fact 1 (minimality of extensions) *Let E_1, E_2 be extensions of T and $E_1 \subseteq E_2$, then $E_1 = E_2$.*

Exercise 6 *Prove Fact 1.*

Theorem 1 (consistency preservation) *$T = (\Gamma, D)$ has an inconsistent extension iff $\Gamma \vdash \bot$.*

As a consequence we get

Fact 2 *If T has an inconsistent extension E, then E is its only extension.*

Exercise 7 *Let $T = (\Gamma, D)$. Prove: if $\Gamma \cup \{\psi_1 \wedge \ldots \wedge \psi_n \wedge \chi : \frac{\varphi:\psi_1,\ldots,\psi_n}{\chi}\}$ is consistent, then T has exactly one extension.*

Theorem 2 (cautious monotony in declaratives) *Let E be an extension of $T = (\Gamma, D)$, then E is an extension of $T' = (\Gamma \cup \Delta, D)$ for every $\Delta \subseteq E$.*

Fact 3 *Every finite process of a default theory T can be extended to a closed process.*

Exercise 8 *Prove Fact 3.*

1.2 Normal default theories

Normal default theories include only rules of the form

$$\frac{\varphi : \chi}{\chi}$$

See, e.g., Exercise 4.

Theorem 3 *Let T be a normal default theory. If Π is a process of T, then Π is successful.*

From Fact 3 and Theorem 3 we get

Theorem 4 (existence of extensions) *If T is a normal default theory, then T has an extension.*

Theorem 5 (monotony in defaults) *Let $T_1 = (\Gamma, D_1)$ and $T_2 = (\Gamma, D_2)$ be normal default theories such that $D_1 \subseteq D_2$. Then each extension E_1 of T_1 is a subset of some extension E_2 of T_2, i.e., $E_1 \subseteq E_2$.*

See Exercise 4: putting both extensions together we get an inconsistent set.

Theorem 6 (orthogonality of extensions) *If a normal default theory T has two different extensions E_1 and E_2, then $E_1 \cup E_2$ is inconsistent.*

Fact 4 *If $T = (\Gamma, D)$ has two different extensions, then $\Gamma \not\vdash \bot$.*

Exercise 9 *Prove Fact 4. Is it valid for general default theories?*

Exercise 10 *Expanding the set of defaults of a normal default theory T does not decrease the number of extensions.*

Hint: Take $T_1 = (\Gamma, D_1)$, $T_2 = (\Gamma, D_2)$ such that $D_1 \subseteq D_2$ and show that two different extensions of T_1 cannot be included in one extension of T_2. Use Theorem 6.

1.2.1 Representation in normal default theories

Definition 4 *Two default theories are equivalent ($T_1 \equiv T_2$) iff T_1 and T_2 have exactly the same extensions.*

Exercise 11 *1. If $\mathsf{Cn}\Gamma_1 = \mathsf{Cn}\Gamma_2$, then $(\Gamma_1, D) \equiv (\Gamma_2, D)$, for every D.*

2. $(\Gamma, D) \equiv (\emptyset, D \cup \{\frac{:}{\varphi} : \varphi \in \Gamma\})$.

3. Let $d_0 = \frac{\varphi:\psi_1,\ldots,\psi_n}{\chi_1 \wedge \chi_2}$, $d_1 = \frac{\varphi:\psi_1,\ldots,\psi_n}{\chi_1}$ and $d_2 = \frac{\varphi:\psi_1,\ldots,\psi_n}{\chi_2}$. Then $(\Gamma, D \cup \{d_0\}) \equiv (\Gamma, D \cup \{d_1, d_2\})$, for every Γ and D.

4. Let $d_0 = \frac{\varphi:\alpha_1 \vee \alpha_2,\psi_1,\ldots,\psi_n}{\chi}$, $d_1 = \frac{\varphi:\alpha_1,\psi_1,\ldots,\psi_n}{\chi}$ and $d_2 = \frac{\varphi:\alpha_2,\psi_1,\ldots,\psi_n}{\chi}$. Then $(\Gamma, D \cup \{d_0\}) \equiv (\Gamma, D \cup \{d_1, d_2\})$, for every Γ and D.

Definition 5 *Let $\mathcal{D}$ be a class of default theories. A theory T is* representable in *$\mathcal{D}$ iff there is a theory $T' \in \mathcal{D}$ such that $T \equiv T'$.*

Exercise 12 *Let $\mathcal{N}$ be a class of all normal default theories. Show that $\mathcal{N}$ does not represent all default theories.*

Hint: Theorem 4 and, e.g., Exercise 1.

Exercise 13 *Let $\mathcal{N}$ be a class of all normal default theories. Show that $\mathcal{N}$ does not represent all default theories possessing extensions, either.*

Hint: Consider a theory $(\emptyset, \{\frac{:\neg p}{q}, \frac{:\neg q}{p}\})$, make its extensions and use Theorem 6.

References

[1] G. Antoniou. *Nonmonotonic Reasoning*. The MIT Press, 1997.